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The Generating Function for the Number of Roots of a Coxeter Group

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Using elementary roots and finite automata, we show that the generating function counting the depths of the roots of a Coxeter group of finite rank is rational.

1. Root Systems

Let (W, R) be a Coxeter system of finite rank $|R|$. Then $W = \langle r \in R : (rs)^{m_{rs}} = 1 \text{ for } r, s \in R\rangle$, where $m_{rr} = 1$ and $m_{rs} \geq 2$ for $r, s \in R$, $r \neq s$ (with $m_{rs} = \infty$ allowed). Let $\Pi = \{\alpha_r : r \in R\}$ be the basis of an $\mathbf{R}$-vector space V. We define a bilinear symmetric form on V by $(\alpha_r, \alpha_s) = -\cos(\pi/m_{rs})$ (and $(\alpha_r, \alpha_s) \leq -1$ for $m_{rs} = \infty$) and let W act on V by $r \cdot v = v - 2(v, \alpha_r)\alpha_r$. Note that this makes the form $(\cdot, \cdot)$ W-invariant.

The set $\Phi = \{w \cdot \alpha_r : w \in W, r \in R\} \subset V$ is called the *root system* of W. Let $\Phi^+ = \{\sum c_r\alpha_r \in \Phi : c_r \geq 0 \text{ for all } r \in R\}$ denote the set of *positive* roots, and put $\Phi^- = -\Phi^+$. It is a fact that $\Phi = \Phi^+ \cup \Phi^-$. An easy consequence is that $r(\Phi^+ \setminus \{\alpha_r\}) = \Phi^+ \setminus \{\alpha_r\}$.

Let $w \in W$. We define $N(w) = \{\alpha \in \Phi^+ : w \cdot \alpha \in \Phi^-\}$. By $\ell(w)$, the *length* of w, we denote the minimal integer n for which there exists an expression $w = r_1 \cdots r_n$ with $r_i \in R$. An expression with $n = \ell(w)$ is called *reduced*.

The following facts are well known and can be found in Humphreys (1990).

Proposition 1.1. *(i) For $w \in W$ and $r \in R$ we have*

$$\ell(wr) = \begin{cases} \ell(w) + 1, & \text{if } w \cdot \alpha_r \in \Phi^+, \\ \ell(w) - 1, & \text{if } w \cdot \alpha_r \in \Phi^-. \end{cases}$$

(ii) For $u, v \in W$ we have $\ell(uv) = \ell(u) + \ell(v)$ iff $N(u) \cap N(v^{-1}) = \emptyset$.

For $\alpha \in \Phi^+$ the depth $\mathrm{dp}(\alpha)$ of α is defined as

$$\mathrm{dp}(\alpha) = \min\{\ell(w) : w \in W, w \cdot \alpha \in \Phi^-\}.$$

Using $r(\Phi^+ \setminus \{\alpha_r\}) = \Phi^+ \setminus \{\alpha_r\}$, it is easily seen that $\mathrm{dp}(\alpha)$ equals the minimum value of $\ell(w) + 1$ taken over all pairs $(w, r) \in W \times R$ satisfying $w \cdot \alpha_r = \alpha$.

For $\alpha \in \Phi$, the reflection r_α with respect to α is defined as $r_\alpha \cdot v = v - 2(v, \alpha)\alpha$. For $w \in W$, $r \in R$ with $w \cdot \alpha_r = \alpha$, we have $r_\alpha = wrw^{-1}$ by the W-invariance of $(\cdot, \cdot)$. Hence $r_\alpha \in W$. The following lemma is Corollary 2.7 of Brink (1994).

Lemma 1.2. *Let $\alpha \in \Phi^+$. Then $\ell(r_\alpha) = 2\,\mathrm{dp}(\alpha) - 1$.*

Proof. Taking $w \in W$, $r \in R$ with $w \cdot \alpha_r = \alpha$ and $\ell(w) + 1 = \mathrm{dp}(\alpha)$, we see that $\ell(r_\alpha) = \ell(wrw^{-1}) \leq 2\ell(w) + 1 = 2\,\mathrm{dp}(\alpha) - 1$.

Now choose a reduced expression $r_\alpha = s_1 \cdots s_n$. Then $\ell(r_\alpha r_\alpha) = \ell(1) < \ell(r_\alpha)$, so by the strong exchange condition (see Humphreys, 1990, Theorem 5.8, p. 117), there is $1 \leq k \leq n$ with

$$1 = r_\alpha r_\alpha = s_1 \cdots s_{k-1} s_{k+1} \cdots s_n.$$

Set $w = s_1 \cdots s_{k-1}$ and $r = s_k$. Then $w^{-1} = s_{k+1} \cdots s_n$. Note that $r_\alpha = wrw^{-1}$ and $\ell(r_\alpha) = 2\ell(w) + 1$. Using $r_\alpha \cdot v = v - 2(v, \alpha)\alpha$ and $r_\alpha \cdot (w \cdot \alpha_r) = (wr) \cdot \alpha_r = -(w \cdot \alpha_r)$, we find that $w \cdot \alpha_r = \pm\alpha$. Since $\ell(wr) > \ell(w)$ we have $w \cdot \alpha_r \in \Phi^+$, so $w \cdot \alpha_r = \alpha$ and hence $\ell(r_\alpha) = 2\ell(w) + 1 \geq 2\,\mathrm{dp}(\alpha) - 1$. □

2. Elementary Roots

Let $\alpha, \beta \in \Phi^+$. We say that α *dominates* β *with respect to* W, written as $\alpha \,\mathrm{dom}_W\, \beta$, iff for all $w \in W$, $w \cdot \alpha \in \Phi^-$ implies $w \cdot \beta \in \Phi^-$. This defines a partial order on Φ^+. We let $\mathcal{E} \subset \Phi^+$ denote the set of minimal elements with respect to this ordering. The elements of $\mathcal{E}$ are called *elementary roots*. Since $r \cdot \alpha_r \in \Phi^-$ and $r(\Phi^+ \setminus \{\alpha_r\}) = \Phi^+ \setminus \{\alpha_r\}$, we have $\Pi \subset \mathcal{E}$. By Lemma 2.2 (iv) of Brink and Howlett (1993), roots $\alpha \neq \beta$ with $\alpha \,\mathrm{dom}_W\, \beta$ satisfy $\mathrm{dp}(\alpha) > \mathrm{dp}(\beta)$. As a consequence, any $\alpha \in \Phi^+$ dominates an elementary root.

Lemma 2.1. *For all $u, v \in W$ we have $\ell(uv) = \ell(u) + \ell(v)$ iff $N(u) \cap N(v^{-1}) \cap \mathcal{E} = \emptyset$.*

Proof. By Proposition 1.1 it suffices to show that if $N(u) \cap N(v^{-1}) \neq \emptyset$, then $N(u) \cap N(v^{-1}) \cap \mathcal{E} \neq \emptyset$. Suppose $\alpha \in N(u) \cap N(v^{-1})$. Then we can find $\beta \in \mathcal{E}$ with $\alpha \,\mathrm{dom}_W\, \beta$. Since $u \cdot \alpha, v^{-1} \cdot \alpha \in \Phi^-$, also $u \cdot \beta, v^{-1} \cdot \beta \in \Phi^-$, so $\beta \in N(u) \cap N(v^{-1}) \cap \mathcal{E}$. □

Brink and Howlett (1993) proved the following fact.

Theorem 2.2. *The set of elementary roots $\mathcal{E}$ is finite.*

Lemma 2.3. *For $w \in W$ and $r \in R$, such that $\ell(wr) > \ell(w)$, we have*

$$N(wr) \cap \mathcal{E} = (r(N(w) \cap \mathcal{E}) \cup \{\alpha_r\}) \cap \mathcal{E}.$$

Proof. Since $N(wr) = rN(w) \cup \{\alpha_r\}$, it is obvious that

$$(r(N(w) \cap \mathcal{E}) \cup \{\alpha_r\}) \cap \mathcal{E} \subset N(wr) \cap \mathcal{E}.$$

To prove the other inclusion, suppose there exists $\alpha \in N(wr) \cap \mathcal{E} = (rN(w) \cup \{\alpha_r\}) \cap \mathcal{E}$ such that $\alpha \notin (r(N(w) \cap \mathcal{E}) \cup \{\alpha_r\}) \cap \mathcal{E}$. Then $\alpha \neq \alpha_r$, so $r \cdot \alpha \in N(w)$ and $r \cdot \alpha \notin \mathcal{E}$. Hence $r \cdot \alpha \,\mathrm{dom}_W\, \beta$ for some $\beta \in \Phi^+$ with $\beta \neq r \cdot \alpha$. Now $r \cdot \beta \in \Phi^+$ would imply $\alpha \,\mathrm{dom}_W\, r \cdot \beta$ (if $w \cdot \alpha \in \Phi^-$, then $wr \cdot (r \cdot \alpha) \in \Phi^-$, so $w \cdot (r \cdot \beta) = wr \cdot \beta \in \Phi^-$). But $\alpha \in \mathcal{E}$ and $\alpha \neq r \cdot \beta$, so $r \cdot \beta \in \Phi^-$ giving $\beta = \alpha_r$ and $r \cdot \alpha \,\mathrm{dom}_W\, \alpha_r$. Since $w \cdot (r \cdot \alpha) \in \Phi^-$, we therefore have $w \cdot \alpha_r \in \Phi^-$, contradicting $\ell(wr) > \ell(w)$. □

For $w \in W$ we define $S(w) = N(w) \cap \mathcal{E}$. Lemma 2.3 can be rephrased as $S(wr) = (rS(w) \cup \{\alpha_r\}) \cap \mathcal{E}$ if $\ell(wr) > \ell(w)$.

LEMMA 2.4. *Let $w \in W$, $r \in R$. We have $w \cdot \alpha_r \in \Phi^+$ and $\mathrm{dp}(w \cdot \alpha_r) = \ell(w) + 1$ iff $\alpha_r \notin S(w)$ and $S(w) \cap rS(w) = \emptyset$.*

PROOF. We start by noting that $w \cdot \alpha_r \in \Phi^+$ iff $\alpha_r \notin S(w)$ iff $\ell(wr) = \ell(w) + 1$. So we must show that for $w \in W$, $r \in R$ with $\alpha_r \notin S(w)$ we have $\mathrm{dp}(w \cdot \alpha_r) = \ell(w) + 1$ iff $S(w) \cap rS(w) = \emptyset$. According to Lemma 1.2, $\mathrm{dp}(w \cdot \alpha_r) = \ell(w) + 1$ is equivalent to $\ell(wrw^{-1}) = 2\ell(w) + 1 = \ell(wr) + \ell(w^{-1})$. By Lemma 2.1, this is equivalent to $N(wr) \cap N(w) \cap \mathcal{E} = \emptyset$. But

$$\begin{aligned} N(wr) \cap N(w) \cap \mathcal{E} = S(wr) \cap S(w) &= ((rS(w) \cap \mathcal{E}) \cup \{\alpha_r\}) \cap S(w) \\ &= rS(w) \cap S(w), \end{aligned}$$

since $S(w) \subset \mathcal{E}$ and $\alpha_r \notin S(w)$. □

3. Rewriting Reduced Expressions

Let R^* denote the free monoid of all words $r_1 \cdots r_n$, $r_i \in R$, $n \geq 0$. There is an obvious projection $\pi : R^* \to W$. For $r, s \in R$, $r \neq s$, let

$$w_{rs} = \underbrace{rsrs\cdots}_{m_{rs}} \in R^*.$$

From $(rs)^{m_{rs}} = 1$ in W, it follows that $\pi(w_{rs}) = \pi(w_{sr})$. Following Tits, substitution of an occurrence w_{rs} by w_{sr} in a word in R^* is called an *elementary simplification of the first kind*. Deletion of an occurrence rr from a word is called an *elementary simplification of the second kind*. Let $\Sigma(\omega) \subset R^*$ denote the set of words that can be obtained from $\omega \in R^*$ by a sequence of elementary simplifications. Clearly all elements of $\Sigma(\omega)$ map to the same element of W under π. Tits (1969) proved the following theorem.

THEOREM 3.1. *Let $\omega_1, \omega_2 \in R^*$. Then $\pi(\omega_1) = \pi(\omega_2)$ iff $\Sigma(\omega_1) \cap \Sigma(\omega_2) \neq \emptyset$.*

Obviously, reduced expressions do not admit elementary simplifications of the second kind.

COROLLARY 3.2. *Let $\omega_1, \omega_2 \in R^*$ be two reduced expressions of an element $w \in W$. Then ω_1 can be transformed into ω_2 by a sequence of elementary simplifications of the first kind.*

Let $\alpha \in \Phi^+$ and choose $(w, r) \in W \times R$ with $w \cdot \alpha_r = \alpha$, $\ell(w) + 1 = \mathrm{dp}(\alpha)$. Suppose there is a second such pair $(v, s) \neq (w, r)$. Note that $w \neq v$. Choosing reduced expressions $w = r_1 \cdots r_n$ and $v = s_1 \cdots s_n$, the word $r_1 \cdots r_n r r_n \cdots r_1$ can be transformed into $s_1 \cdots s_n s s_n \cdots s_1$ by elementary simplifications of the first kind. If such simplifications are applied to the first n or the last n positions, the original expression is transformed into a word of the form $\omega_1 r \omega_2$ with $\pi(\omega_1) = w$, $\pi(\omega_2) = w^{-1}$. So at some point we obtain reduced expressions $r'_1 \cdots r'_n$ for w and $r''_n \cdots r''_1$ for w^{-1}, such that in R^* we have $r'_1 \cdots r'_n r r''_n \cdots r''_1 = r'_1 \cdots r'_k w_{pq} r''_l \cdots r''_1$ for some $1 \leq k, l \leq n$ and $p, q \in R$. Suppose $k \neq l$. Then we can either replace $r'_1 \cdots r'_n$ by $r''_1 \cdots r''_n$, or $r''_n \cdots r''_1$ by $r'_n \cdots r'_1$, to obtain an expression of r_α of length $\ell(r_\alpha)$, with a number of alternating p and q exceeding m_{pq}, contradicting reducedness. Hence $k = l$, m_{pq} is odd, and the simplification $w_{pq} \mapsto w_{qp}$ transforms wrw^{-1} into an expression utu^{-1}, where $\{r, t\} = \{p, q\}$ and $u = w(tr)^{(m_{rt}-1)/2}$.

Note that the pair $(u,t) \in W \times R$ is not necessarily equal to (v,s), but it satisfies $u \cdot \alpha_t = \alpha$ and $\ell(u) + 1 = \mathrm{dp}(\alpha)$, and, after a finite number of such transitions, we will arrive at (v,s).

We now consider all pairs $(w,r) \in W \times R$ with $w \cdot \alpha_r = \alpha$ and $\mathrm{dp}(\alpha) = \ell(w) + 1$. We construct a graph G_α by taking these pairs as vertices and letting $\{(w,r),(v,s)\}$ $(r \neq s)$ be an edge if $v = w(sr)^{(m_{rs}-1)/2}$.

Lemma 3.3. *The graph G_α is a tree.*

Proof. Corollary 3.2 and the discussion following it imply that G_α is connected. We have to show that G_α contains no cycles. Assuming the contrary, let

$$(w_0,r_0),(w_1,r_1),\ldots,(w_n,r_n) = (w_0,r_0) \qquad (n > 0)$$

be a cycle in G_α of minimal length. Note that $r_k \neq r_{k+2}$ for $0 \leq k \leq n-2$. Now

$$w_0 = w_n = w_0(r_1r_0)^{(m_{r_0r_1}-1)/2} \cdots (r_nr_{n-1})^{(m_{r_{n-1}r_n}-1)/2},$$

so

$$(r_1r_0)^{(m_{r_0r_1}-1)/2} \cdots (r_nr_{n-1})^{(m_{r_{n-1}r_n}-1)/2} = 1.$$

Let $\omega \in R^*$ denote the expression on the left. We claim that ω is reduced, contradicting the existence of the cycle. To prove this claim, we use an argument based on Theorem 3.1. An alternative argument can be found in Section 2 in Brink (1996).

We consider elementary simplifications of the type $rs \mapsto sr$. Writing $\omega = s_1 \cdots s_N$ with $s_i \in \{r_1,\ldots,r_n\}$, these simplifications induce transpositions $(i\;\; i+1)$ of $\{1,\ldots,N\}$. Let ω' denote a word obtained from ω by a sequence of such simplifications. Composing the corresponding transpositions, we find a permutation σ satisfying $\omega' = s_{\sigma^{-1}(1)} \cdots s_{\sigma^{-1}(N)}$.

Choose $i < j$ with $s_i = s_j$. Observe that $j - i \geq 2$, and that both s_{i+1} and s_{j-1} do not commute in W with $s_i = s_j$. Since ω' is obtained by transposing consecutive commuting elements, we have $\sigma(i) < \sigma(i+1) < \sigma(j)$ and $\sigma(i) < \sigma(j-1) < \sigma(j)$. We therefore have $\sigma(j) - \sigma(i) \geq 2$ if $j - i = 2$, and $\sigma(j) - \sigma(i) \geq 3$ if $j - i \geq 3$.

It now easily follows that any elementary simplification applicable to ω' is of the form $rs \mapsto sr$. Indeed, the preceding paragraph shows that $s_{\sigma^{-1}(i)} = s_{\sigma^{-1}(j)}$ $(i \neq j)$ implies $|i - j| \geq 2$, so elementary simplifications of the second kind are not possible. To exclude the other simplifications, note that if $s_{\sigma^{-1}(i)}s_{\sigma^{-1}(i+1)} \cdots s_{\sigma^{-1}(j)} = w_{rs}$ with $j - i \geq 2$, then, also by the above, $\sigma^{-1}(i),\ldots,\sigma^{-1}(j)$ are consecutive integers, implying that w_{rs} already occurs in ω, which is not the case. Now our claim follows by Theorem 3.1. $\square$

4. A Finite Automaton as Counting Machine

Let $\prec$ be an ordering on R. In Section 3 of Brink and Howlett (1993), the finiteness of $\mathcal{E}$ is used to construct a finite automaton accepting lexicographically minimal reduced words. We shall now construct a finite automaton accepting the same language, but having some extra information in the states enabling us to count particular kinds of group elements of W. The states $\mathcal{S}$ of the automaton will be the pairs (S,T) of subsets of $\mathcal{E}$ (the accept states) together with a state *reject*. The initial state is $(\emptyset,\emptyset)$. The transition function $\mu : \mathcal{S} \times R \to \mathcal{S}$ is given by (i) $\mu(\mathit{reject}, r) = \mathit{reject}$ for $r \in R$,

(ii) $\mu((S,T),r) = \textit{reject}$ if $\alpha_r \in S$, (iii) $\mu((S,T),r) = (S',T')$ if $\alpha_r \notin S$, where

$$S' = (r(S) \cup \{\alpha_r\} \cup \{r \cdot \alpha_s : s \in R, s \prec r\}) \cap \mathcal{E}$$

and

$$T' = (r(T) \cup \{\alpha_r\}) \cap \mathcal{E}.$$

Identification of states with equal sets S would give the automaton defined in Brink and Howlett (1993), with fewer states, but accepting the same language. So any $w \in W$ has a unique reduced expression that is accepted. Using Lemma 2.3 we see that if $w \in W$ is accepted in state (S,T), we have $T = S(w)$.

It is not difficult to see that the generating function $f_i(t)$ counting the lengths of the words accepted in state i is rational. In fact, if A is the transition matrix of the automaton (indexed by the accept states and having as entry (i,j) the number of $r \in R$ with $\mu(j,r) = i$), the vector $f = (f_i(t))$ of generating functions where i runs over the accept states is the unique vector for which $(I - tA)f$ has an entry 1 for the initial state and 0 for the others.

For $r \in R$, let $\mathcal{S}_r \subset \mathcal{S}$ denote the states (S,T) with $\alpha_r \notin T$ and $T \cap r(T) = \emptyset$. Let n_α denote the number of vertices of G_α. According to Lemma 2.4, a pair $(w,r) \in W \times R$ occurs as a vertex in some G_α iff $\alpha_r \notin S(w)$ and $S(w) \cap rS(w) = \emptyset$, so iff the unique state at which w is accepted is included in $\mathcal{S}_r$. Therefore,

$$\sum_{r \in R} \sum_{i \in \mathcal{S}_r} t f_i(t) = \sum_{\alpha \in \Phi^+} n_\alpha t^{\mathrm{dp}(\alpha)}. \tag{4.1}$$

Now we count the edges of the graphs G_α. Fix $r, s \in R$ with $r \prec s$. Suppose $m_{rs} = 2k+1$ is odd. We consider all pairs $\{(w,r),(v,s)\}$ that occur as edge in some G_α. From the definition of the G_α, it follows that there is a bijection between these pairs and elements $u \in W$ satisfying

$$\ell(u\underbrace{\cdots rs}_{k}) = \ell(u\underbrace{\cdots sr}_{k}) = \ell(u) + k \tag{4.2}$$

and

$$\mathrm{dp}(u\underbrace{\cdots rs}_{k}\cdot\alpha_r) = \ell(u) + k + 1. \tag{4.3}$$

So each such u corresponds to an edge in some G_α with $\mathrm{dp}(\alpha) = \ell(u) + k + 1 = \ell(u) + (m_{rs} + 1)/2$.

Note that $\ell(ur) > \ell(u)$ iff $\alpha_r \notin S(u)$, and that $S(ur) = (rS(u) \cup \{\alpha_r\}) \cap \mathcal{E}$ if $\ell(ur) > \ell(u)$. Repeating this, we see that condition (4.2) amounts to a condition on $S(u)$, and that if (4.2) holds, the set $S(u\underbrace{\cdots rs}_{k})$ is determined by $S(u)$. Then by Lemma 2.4 it follows that, given the truth of (4.2), condition (4.3) can also be translated into a condition on $S(u)$. We conclude that there is a set $\mathcal{T}_{r,s}$ of accept states having the property that $u \in W$ satisfies (4.2) and (4.3) iff u is accepted in a state contained in $\mathcal{T}_{r,s}$.

If m_{rs} is even, there are no edges $\{(w,r),(v,s)\}$, and we set $\mathcal{T}_{r,s} = \emptyset$.

Since each G_α is a tree, it has $n_\alpha - 1$ edges. Hence

$$\sum_{r \prec s} \sum_{i \in \mathcal{T}_{r,s}} t^{(m_{rs}+1)/2} f_i(t) = \sum_{\alpha \in \Phi^+} (n_\alpha - 1) t^{\mathrm{dp}(\alpha)}. \tag{4.4}$$

Combining (4.1) and (4.4), we obtain the following result.

THEOREM 4.1. *The generating function counting the depths of the roots of a Coxeter system of finite rank is rational. In the notation of this section, it is given by*

$$\sum_{\alpha \in \Phi^+} t^{\mathrm{dp}(\alpha)} = \sum_{r \in R} \sum_{i \in \mathcal{S}_r} t f_i(t) - \sum_{r \prec s} \sum_{i \in \mathcal{T}_{r,s}} t^{(m_{rs}+1)/2} f_i(t). \tag{4.5}$$

5. Remarks

5.1. EXAMPLES

The finite automaton, and hence also the generating functions, can be effectively computed. We illustrate this by two examples.

Given a Coxeter system (W, R) of finite rank, there is an easy method to construct the set of elementary roots $\mathcal{E}$. Start with the set $\Pi = \{\alpha_r : r \in R\} \subset \mathcal{E}$, and keep adding roots $r \cdot \alpha$ with $r \in R$, $\alpha \in \Phi^+$ already known to be elementary, and satisfying $-1 < (\alpha, \alpha_r) < 0$. The correctness of this method follows immediately from Lemma 3.10 from Brink (1994).

As an example, for the affine group $\tilde{A}_2$ of rank 3 we obtain

$$\mathcal{E} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_1 + \alpha_3, \alpha_2 + \alpha_3\},$$

where $\Pi = \{\alpha_1, \alpha_2, \alpha_3\}$.

The states of the finite state automaton used for counting are the pairs (S, T) with $S, T \subset \mathcal{E}$ (together with *reject*). The interesting part is the connected component containing the initial state $(\emptyset, \emptyset)$. These states (S, T) are easily seen to satisfy $T \subset S$, so the number of them is bounded from above by $3^{|\mathcal{E}|}$. In practice, however, it seems that this bound is rather bad. For $\tilde{A}_2$ we obtain a component consisting of 13 accept states.

Given the transition matrix A of the component, finding the vector $f = (f_i(t))$ of generating functions for the accept states i is a straightforward calculation, as described in Section 4. For $\tilde{A}_2$, we have

$$\sum_i f_i(t) = \frac{1 + t + t^2}{(1-t)^2},$$

which is no surprise since both sides count the lengths of the group elements of $\tilde{A}_2$. (The right-hand side can be computed independently by using either Section 5.12 or Section 8.9 of Humphreys (1990).)

Working out equation (4.5) for $\tilde{A}_2$, we obtain

$$\sum_{\alpha \in \Phi^+} t^{\mathrm{dp}(\alpha)} = \frac{3t}{1-t}.$$

This result can also be derived from facts on affine root systems.

As a second example, we take the Coxeter group of rank 4 with matrix (m_{rs}) given by

$$\begin{pmatrix} 1 & 3 & 2 & 2 \\ 3 & 1 & 3 & 3 \\ 2 & 3 & 1 & 3 \\ 2 & 3 & 3 & 1 \end{pmatrix}.$$

Writing $\Pi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, the set of elementary roots turns out to be

$$\mathcal{E} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_2 + \alpha_4, \alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_4\}.$$

The component of the finite automaton containing the initial state consists of 209 accept states. Equation (4.5) now takes the form

$$\sum_{\alpha \in \Phi^+} t^{\mathrm{dp}(\alpha)} = \frac{4t - 4t^2 + t^3 - 4t^4 + t^5}{1 - 2t + t^2 - t^3 + t^4}.$$

5.2. Counting reflections

Since $\alpha \mapsto r_\alpha$ is a bijection from Φ^+ to the set of reflections of W, and $\ell(r_\alpha) = 2\,\mathrm{dp}(\alpha) - 1$ by Lemma 1.2, the generating function counting the depths can easily be transformed into the generating function counting the lengths of all reflections. Indeed, if $P(t) = \sum_{\alpha \in \Phi^+} t^{\mathrm{dp}(\alpha)}$, then $P(t^2)/t = \sum_{r_\alpha \in W} t^{\ell(r_\alpha)}$.

It is clear that conjugates of reflections are themselves reflections, and that each reflection is conjugate to an element of R. There is an easy way to partition R into conjugacy classes: they are the connected components of the graph with vertices R and edges $\{r, s\}$ for $r \neq s$ with m_{rs} finite and odd.

Section 4 can easily be adapted to count the reflections in a single conjugacy class: in equation (4.5), sum only over r, s belonging to the corresponding graph component. (Note that $\mathcal{T}_{r,s}$ is empty for r, s not in the same component.)

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